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Odd Poisson brackets and the fermionic hierarchy of Becker and Becker

I N McArthur
Department of Physics, The University of Western Australia, Nedlands, Australia 6009

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Abstract. A new fermionic extension of the KdV hierarchy was recently found by Becker and Becker in relation to two-dimensional quantum supergravity, and it was shown that the hierarchy possesses a supersymmetry which has been exploited to prove the integrability of the hierarchy by Figueroa-O'Farrill and Stanciu. Here, a larger group of fermionic symmetries of the new hierarchy is demonstrated and the bi-Hamiltonian structure is expressed in terms of odd Poisson brackets which are related to the antibracket of the Batalin–Vilkovisky formalism.

1. Introduction

One of the more surprising revelations about two-dimensional quantum gravity which has arisen with the advent of matrix models is the relation to the KdV hierarchy of nonlinear partial differential equations. In the quest to identify the corresponding structure underlying two-dimensional quantum supergravity [1], a new fermionic extension of the KdV hierarchy was recently revealed by Becker and Becker [2]. The hierarchy was identified as being supersymmetric, and has been analysed further by Figueroa-O'Farrill and Stanciu [3] by drawing on the supersymmetry. In this paper, we point out that the hierarchy has a larger fermionic symmetry than supersymmetry, and that its Hamiltonian structure resembles the antibracket in the Batalin–Vilkovisky [4] approach to theories with gauge invariance.

2. The hierarchy of Becker and Becker

The KdV hierarchy of nonlinear partial differential equations takes the form

$$\frac{\partial u}{\partial t_n} = \mathrm{D}R_{n+1} \qquad n \geqslant 0 \tag{1}$$

where $u = u(x = t_0, t_1, t_2, \ldots)$, $\mathrm{D} = \partial/\partial x$ and the Gelfand–Dikii polynomials $R_{n+1}(x)$ are defined by the recursion relation $\mathrm{D}R_{n+1}(x) = (\mathrm{D}^2 + 4u(x)\mathrm{D} + 2(\mathrm{D}u(x)))R_n(x)$ with $R_0(x) = \frac{1}{2}$. Defining the differential operator $\mathbf{O}_{n+1}(x)$ by

$$\delta R_{n+1}(x) = \mathbf{O}_{n+1}(x) \cdot \delta u(x) \tag{2}$$

for arbitrary infinitesimal variations δu of u, the fermionic extension of the KdV hierarchy considered in [2] involves a Grassmann-odd function $\tau(x, t_1, t_2, \ldots)$ satisfying the flow equations

$$\frac{\partial \tau}{\partial t_n} = \mathbf{O}_{n+1} \cdot \mathrm{D}\tau \qquad n \geqslant 0. \tag{3}$$

It was pointed out in [2] that the hierarchy of equations (1) and (3) is invariant under the supersymmetry transformations $\delta_S u = \epsilon D\tau$, $\delta_S \tau = \epsilon u$ (with ϵ a Grassmann parameter). By use of techniques adapted from the analysis of other supersymmetric extensions of the KdV hierarchy, this new fermionic extension of the KdV hierarchy was shown to be integrable in [3].

In fact, the hierarchy (1), (3) possesses the much larger symmetry

$$\delta u = 0 \qquad \delta\tau = \epsilon u \qquad \tilde{\delta} u = \tilde{\epsilon} D\tau \qquad \tilde{\delta}\tau = 0$$

where ϵ and $\tilde{\epsilon}$ are independent Grassmann parameters. The supersymmetry is the special case $\epsilon = \tilde{\epsilon}$.

To verify the symmetry of the hierarchy under the transformation δ, it is necessary to check that $u + \delta u$ and $\tau + \delta\tau$ fulfil the equations of the hierarchy (1), (3) if u and τ do. This follows trivially using $\mathbf{O}_{n+1}(x) \cdot Du(x) = DR_{n+1}(x)$ from (2).

The symmetry of the hierarchy under the transformations $\tilde{\delta}$ is less trivial, and follows from $(\tilde{\delta}\mathbf{O}_{n+1}(x))\cdot D\tau(x) = 0$. This can be proved by noting $\tilde{\delta}_1\tilde{\delta}_2 R_{n+1}(x) = -\tilde{\epsilon}_2(\tilde{\delta}_1\mathbf{O}_{n+1}(x))\cdot D\tau(x)$. The left-hand side of this expression vanishes, since it is of the form

$$-\tilde{\epsilon}_2\tilde{\epsilon}_1 \sum_{i\geqslant 0}\sum_{j\geqslant 0} \frac{\partial^2 R_{n+1}(x)}{\partial u^{(i)}(x)\partial u^{(j)}(x)} (D^{i+1}\tau(x))(D^{j+1}\tau(x))$$

(where $u^{(i)} = D^i u$), and the partial derivative is symmetric under $i \to j$ while $(D^{i+1}\tau)(D^{j+1}\tau)$ is antisymmetric.

As already noted, supersymmetry is equivalent to symmetry under the transformation $\delta + \tilde{\delta}$ in the case $\epsilon = \tilde{\epsilon}$. This also forces a restriction on the scaling dimension of τ. The equations (1), (3) are invariant under the rescalings $\partial/\partial t_n \to \lambda^{2n+1}\partial/\partial t_n$, $u \to \lambda^2 u$, while there is no restriction on the scaling properties of τ as the equations are linear in τ. Symmetry of the equations with respect to the transformations δ and $\tilde{\delta}$ require that ϵ and $\tilde{\epsilon}$ scale as $\lambda^{\alpha-2}\epsilon$ and $\lambda^{1-\alpha}\tilde{\epsilon}$ if τ scales as $\lambda^{\alpha}\tau$. If the hierarchies of equations are required to be supersymmetric, then $\epsilon = \tilde{\epsilon}$, so that $\alpha = \frac{3}{2}$ and there is a restriction on the scaling dimension of τ. In terms of the application of this hierarchy to two-dimensional quantum supergravity [2], τ has a scaling dimension $\alpha = 0$, since $\kappa^2\partial^2 \ln Z_S = u - 2\tau D^2\tau$, where Z_S is the partition function. This suggests that for applications of the hierarchy (1), (3) to quantum supergravity, the supersymmetry is not a natural symmetry.

In the rest of the paper, the bi-Hamiltonian structure for the new fermionic hierarchy expressed in [3] in terms of a supersymmetric construction will be cast in the form of an odd Poisson bracket structure reminiscent of that found in the antifield formalism of Batalin and Vilkovisky. This removes the emphasis placed on the supersymmetry in the analysis of the hierarchy in [3], and may be a more appropriate setting in which to interpret the much larger symmetry of the hierarchy.

3. Odd Poisson brackets in a finite-dimensional setting

Before dealing with the new fermionic hierarchy, we review in this section the odd Poisson bracket which can be defined on the cotangent bundle to a finite-dimensional manifold with symplectic structure [5], and which is related to the setting for the geometrical interpretation of the antibracket formalism in gauge theories [6].

If M is an even-dimensional manifold with symplectic structure $\omega(x) = \omega_{ij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j$, then this determines a Poisson bracket defined by

$$\{f(x), g(x)\}_{\mathrm{PB}} = \sum_{i,j} \frac{\partial f(x)}{\partial x^i} \omega^{ij}(x) \frac{\partial g(x)}{\partial x^j} \tag{4}$$

where f and g are functions on M. In particular, for the coordinate functions, $\{x^i, x^j\}_{\mathrm{PB}} = \omega^{ij}(x)$.

Introducing fermionic (or Grassmann-odd) coordinates θ_i in addition to the x^i (where the θ_i correspond to tangent vectors $\partial/\partial x^i$ to M in a geometric interpretation of the antibracket [6]), the odd Poisson bracket $\{\,,\,\}_{\mathrm{AB}}$ (analogue of the antibracket) is defined by†

$$\{x^i, x^j\}_{\mathrm{AB}} = 0 \qquad \{x^i, \theta_j\}_{\mathrm{AB}} = \delta^i_j, \qquad \{\theta_i, \theta_j\}_{\mathrm{AB}} = 0.$$

In terms of coordinates $\theta^i = \sum_j \omega^{ij}(x)\theta_j$ [5], this becomes

$$\{x^i, x^j\}_{\mathrm{AB}} = 0 \qquad \{x^i, \theta^j\}_{\mathrm{AB}} = \omega^{ij}(x) = \{\theta^i, x^j\}_{\mathrm{AB}} \qquad \{\theta^i, \theta^j\}_{\mathrm{AB}} = \sum_k \frac{\partial \omega^{ij}(x)}{\partial x^k}\theta^k. \tag{5}$$

(Note: it is necessary to use the closure of ω, $0 = \partial\omega_{ij}/\partial x^k +$ (cyclic permutations), to show the last of these.)

Given an *ordinary* Hamiltonian flow (i.e. one determined by the even Poisson bracket $\{\,,\,\}_{\mathrm{PB}}$)

$$\partial x^i/\partial t = \{x^i, H(x)\}_{\mathrm{PB}} = \sum_k \omega^{ik}(x)\partial H(x)/\partial x^k \tag{6}$$

for some Hamiltonian $H(x)$, then it is possible to form a natural extension to a flow for the coordinates θ^i. Replacing x^i in (6) by $X^i = x^i + \xi\theta^i$ (with ξ a Grassmann parameter) and taking the coefficients of 1 and ξ gives the system of equations

$$\frac{\partial x^i}{\partial t} = \sum_k \omega^{ik}(x)\frac{\partial H(x)}{\partial x^k}$$

$$\frac{\partial \theta^i}{\partial t} = \sum_{k,l}\Big(\frac{\partial \omega^{ik}(x)}{\partial x^l}\frac{\partial H(x)}{\partial x^k} + \omega^{ik}(x)\frac{\partial^2 H(x)}{\partial x^k \partial x^l}\Big)\theta^l.$$

It is readily verified that this system of equations can be put in Hamiltonian form with respect to the odd Poisson bracket as [5]

$$\partial x^i/\partial t = \{x^i, \hat{H}(x)\}_{\mathrm{AB}}$$

$$\partial \theta^i/\partial t = \{\theta^i, \hat{H}(x)\}_{\mathrm{AB}}$$

where $\hat{H}(x) = \Sigma_k(\partial H(x)/\partial x^k)\theta^k$ is the coefficient of ξ in the expansion of $H(X)$. Note that $\hat{H}(x)$ is of necessity Grassmann-odd since the odd Poisson bracket changes the Grassmann grading of functions.

† The precise definition is

$$\{f, g\}_{\mathrm{AB}} = \sum_k \left(\frac{\partial_{\mathrm{R}} f}{\partial x^k}\frac{\partial_{\mathrm{L}} g}{\partial \theta_k} - \frac{\partial_{\mathrm{R}} f}{\partial \theta_k}\frac{\partial_{\mathrm{L}} g}{\partial x^k}\right)$$

where ∂_{L} and ∂_{R} denote left and right derivatives respectively, and f and g denote functions of the x^i and θ_i.

4. The Hamiltonian structure of the new fermionic hierarchy

In this section, it will be shown that the bi-Hamiltonian structure for the hierarchy (1), (3) demonstrated in [3] (from which integrability follows) can be obtained by an extension of the above formalism to the infinite-dimensional case. This avoids the emphasis placed on the supersymmetry in [3], which is not well adapted to applications to quantum supergravity.

We begin with the purely *bosonic* KdV hierarchy (1), which can be expressed in the bi-Hamiltonian form

$$\partial u(x)/\partial t_n = \{u(x), H_{n+1}\}_{\mathrm{PB1}} = \{u(x), H_n\}_{\mathrm{PB2}} \tag{7}$$

with $H_n = (4n+2)^{-1} \int \mathrm{d}x\, R_{n+1}(x)$ and the Poisson brackets

$$\{u(x), u(x')\}_{\mathrm{PB1}} = \mathrm{D}\delta(x, x') \tag{8}$$

$$\{u(x), u(x')\}_{\mathrm{PB2}} = (\mathrm{D}^3 + 4u(x)\mathrm{D} + 2(\mathrm{D}u(x)))\delta(x, x'). \tag{9}$$

The integrability of the KdV hierarchy is a consequence of the existence of this bi-Hamiltonian structure. For functionals $F[u]$ and $G[u]$ of u (i.e. $F[u] = \int \mathrm{d}x\, f(x)$ where $f(x)$ is a polynomial in $u(x)$ and its derivatives, and similarly for $G[u]$),

$$\{F[u], G[u]\}_{\mathrm{PB}} = \int \mathrm{d}x \int \mathrm{d}x' \, \frac{\delta F[u]}{\delta u(x)} \{u(x), u(x')\}_{\mathrm{PB}} \frac{\delta G[u]}{\delta u(x')}.$$

Comparing with (4), we see that in the infinite-dimensional case, the analogue of the index i is x, the analogue of x^i is $u(x)$, and the analogue of $\omega^{ij}(x) = \{x^i, x^j\}_{\mathrm{PB}}$ is $\{u(x), u(x')\}_{\mathrm{PB}}$. So, introducing a Grassmann-odd function $\mathrm{D}\tau(x)$ to correspond to θ^i, the infinite-dimensional analogue of the odd Poisson bracket (5) is

$$\begin{aligned}
&\{u(x), u(x')\}_{\mathrm{AB}} = 0 \\
&\{u(x), \mathrm{D}\tau(x')\}_{\mathrm{AB}} = \{u(x), u(x')\}_{\mathrm{PB}} = \{\mathrm{D}\tau(x), u(x')\}_{\mathrm{AB}} \\
&\{\mathrm{D}\tau(x), \mathrm{D}\tau(x')\}_{\mathrm{AB}} = \int \mathrm{d}x'' \, \frac{\delta\{u(x), u(x')\}_{\mathrm{PB}}}{\delta u(x'')} \mathrm{D}\tau(x'').
\end{aligned}$$

There will be two such odd Poisson brackets, one for each of the Poisson brackets (8) and (9). They take the form

$$\begin{aligned}
&\{u(x), u(x')\}_{\mathrm{AB1}} = 0 \\
&\{u(x), \mathrm{D}\tau(x')\}_{\mathrm{AB1}} = \mathrm{D}\delta(x, x') = \{\mathrm{D}\tau(x), u(x')\}_{\mathrm{AB1}} \\
&\{\mathrm{D}\tau(x), \mathrm{D}\tau(x')\}_{\mathrm{AB1}} = 0
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\{u(x), u(x')\}_{\mathrm{AB2}} = 0 \\
&\{u(x), \mathrm{D}\tau(x')\}_{\mathrm{AB2}} = (\mathrm{D}^3 + 4u(x)\mathrm{D} + 2(\mathrm{D}u(x)))\delta(x, x') \\
&\{\mathrm{D}\tau(x), u(x')\}_{\mathrm{AB2}} = (\mathrm{D}^3 + 4u(x)\mathrm{D} + 2(\mathrm{D}u(x)))\delta(x, x') \\
&\{\mathrm{D}\tau(x), \mathrm{D}\tau(x')\}_{\mathrm{AB2}} = (4(\mathrm{D}\tau(x))\mathrm{D} + 2(\mathrm{D}^2\tau(x)))\delta(x, x').
\end{aligned} \tag{11}$$

In [3], the vital observation was made that the hierarchy (1), (3) can be expressed by introducing a 'superfield' $U = u + \xi \mathrm{D}\tau$ (with ξ a Grassmann parameter) and taking the coefficients of 1 and ξ respectively in

$$\partial U/\partial t_n = \mathrm{D}R_{n+1}(U).$$

Denoting by $\hat{H}_n$ the coefficient of ξ in the expansion of $H_n(U) = (4n+2)^{-1}\int \mathrm{d}x\, R_{n+1}(U)$, analogy with the finite-dimensional case would suggest that the new fermionic hierarchy can be expressed in bi-Hamiltonian form with respect to the odd Poisson bracket as

$$\frac{\partial u(x)}{\partial t_n} = \{u(x), \hat{H}_{n+1}\}_{AB1} = \{u(x), \hat{H}_n\}_{\mathrm{AB2}} \tag{12}$$

$$\frac{\partial \mathrm{D}\tau(x)}{\partial t_n} = \{\mathrm{D}\tau(x), \hat{H}_{n+1}\}_{\mathrm{AB1}} = \{\mathrm{D}\tau(x), \hat{H}_n\}_{\mathrm{AB2}}. \tag{13}$$

This is verified explicitly in the appendix. In fact, this is the bi-Hamiltonian structure for the new fermionic hierarchy obtained in [3] (where it is written implicitly in superfield form and not in terms of Poisson brackets).

5. Conclusion

In this paper, we have shown that the new fermionic hierarchy of integrable equations (1), (3) found by Becker and Becker [2] in relation to two-dimensional quantum supergravity has a much larger symmetry than the supersymmetry used in [3] to analyse its integrability. It is also shown that the bi-Hamiltonian structure is related to an infinite-dimensional analogue of the odd Hamiltonian structure of [5], which is in turn related to the Batalin–Vilkovisky antibracket used for the analysis of theories with gauge invariances. It would be of interest to try and relate the odd Poisson bracket to the larger group of fermionic symmetries of the new fermionic hierarchy.

Appendix

Here, we verify explicitly that the Hamiltonian system (12), (13) is equivalent to the new fermionic hierarchy (1), (3). Some preliminary results are required.

The first is

$$\int \mathrm{d}x\, \mathbf{O}_{n+1}(x) \cdot f(x) = (4n+2)\int \mathrm{d}x\, f(x) R_n(x) \tag{14}$$

for an arbitrary function $f(x)$ (or equivalently, $\mathbf{O}_{n+1}(x)^* \cdot 1 = (4n+2)R_n(x)$, where $*$ denotes the adjoint operator). This follows from consideration of $\delta \int \mathrm{d}x\, R_{n+1}(x)$ with $\delta u(x) = f(x)$. The left-hand side of (14) arises using (2), while the right-hand side is the usual result $\delta R_{n+1}(x)/\delta u(x) = (4n+2)R_n(x)$ for the variational derivative of the Gelfand–Dikii polynomials [7] (note that the normalization of the Gelfand–Dikii polynomials used here differs from that in [7]).

The second is that

$$\mathbf{O}_n(x)^* = \mathbf{O}_n(x). \tag{15}$$

This follows from the equation $(\delta_1\delta_2 - \delta_2\delta_1)\int \mathrm{d}x\, R_n(x) = 0$ for arbitrary infinitesimal variations $\delta_1 u$ and $\delta_2 u$ of u, with the use of (2) and (14).

Finally, an expression for $\hat{H}_n$ is required. From the definition $H_n + \xi\hat{H}_n = (4n+2)^{-1}\int \mathrm{d}x\, R_{n+1}(u(x) + \xi \mathrm{D}\tau(x))$, it follows that $\hat{H}_n = (4n+2)^{-1}\int \mathrm{d}x'\, \mathbf{O}_{n+1}(x') \cdot \mathrm{D}\tau(x')$, or after application of (14),

$$\hat{H}_n = \int \mathrm{d}x'\, (\mathrm{D}\tau(x'))R_n(x'). \tag{16}$$

We are now in a position to verify (12). Using (16),

$$\{u(x), \hat{H}_{n+1}\}_{\mathrm{AB1}} = \int \mathrm{d}x'\, \{u(x), \mathrm{D}\tau(x')\}_{\mathrm{AB1}} R_{n+1}(x').$$

Substitution of (11) yields

$$\{u(x), \hat{H}_{n+1}\}_{\mathrm{AB1}} = \mathrm{D}R_{n+1}(x)$$

which is equivalent to $\partial u(x)/\partial t_n$ by (1).

Similarly, using (16) and (2),

$$\{\mathrm{D}\tau(x), \hat{H}_{n+1}\}_{\mathrm{AB1}} = \int \mathrm{d}x'\, (\mathbf{O}_{n+1}(x') \cdot \{\mathrm{D}\tau(x), u(x')\}_{\mathrm{AB1}})\mathrm{D}\tau(x').$$

Substitution of (11) and use of (15) shows that the right-hand side is equal to $\mathrm{D}\mathbf{O}_{n+1}(x) \cdot \mathrm{D}\tau(x)$, which is the required result $\partial D\tau/\partial t_n$ by (3).

Moving to the second odd Poisson structure,

$$\begin{aligned}\{u(x), \hat{H}_n\}_{\mathrm{AB2}} &= \int \mathrm{d}x' \{u(x), \mathrm{D}\tau(x')\}_{\mathrm{AB2}} R_n(x') \\ &= (\mathrm{D}^3 + 4u(x)\mathrm{D} + 2(\mathrm{D}u(x)))R_n(x)\end{aligned}$$

(using (12)), which is equal to $\mathrm{D}R_{n+1}(x)$ via the recursion relation satisfied by the Gelfand–Dikii polynomials.

The case $\{\mathrm{D}\tau(x), \hat{H}_n\}_{\mathrm{AB2}}$ is slightly more complicated. Using (16),

$$\begin{aligned}\{\mathrm{D}\tau(x), \hat{H}_n\}_{\mathrm{AB2}} = &\int \mathrm{d}x'\, \{\mathrm{D}\tau(x), \mathrm{D}\tau(x')\}_{\mathrm{AB2}} R_n(x') \\ &+ \int \mathrm{d}x'\, (\mathbf{O}_n(x').\{\mathrm{D}\tau(x), u(x')\}_{\mathrm{AB2}})\mathrm{D}\tau(x').\end{aligned}$$

Applying (12) and (15), the right-hand side is equivalent to

$$(4(\mathrm{D}\tau(x))\mathrm{D} + 2(\mathrm{D}^2\tau(x)))R_n(x) + (\mathrm{D}^3 + 4u(x)\mathrm{D} + 2(\mathrm{D}u(x)))\mathbf{O}_n(x) \cdot \mathrm{D}\tau(x).$$

By taking the coefficient of ξ in the recursion relation $\mathrm{D}R_{n+1}(U) = (\mathrm{D}^3 + 4U\mathrm{D} + 2(\mathrm{D}U))R_n(U)$ with $U = u + \xi\mathrm{D}\tau$, this is seen to be equivalent to $\mathrm{D}\mathbf{O}_{n+1}(x) \cdot \mathrm{D}\tau(x)$, which is $\partial \mathrm{D}\tau(x)/\partial t_n$ by (3).

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